Stability of Gyroscopes Orbiting in a Gravitational Field

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When the output axis of a single-degree-of-freedom gyroscope is fixed in a rotating satellite, and the gimbal ring is connected to the satellite by means of a spring and damper, the spin axis (rotor axis) of the gyro, if initially at rest relative to the satellite, may tend to return to its initial position subsequent to a small disturbance, in which case the system is said to be "stable," or it may seek some position other than the initial one, a state of affairs termed "unstable." As might be expected, the motion of the satellite, the physical characteristics of the gyro, such as inertia properties, spring constants, etc., and the spin rate of the rotor all have an effect on stability, and the relationship between stability and the values of various system parameters is rather complex. The present paper consists of a detailed exploration of this relationship.

Introduction

STABILITY of a single-axis gyro mounted on a vehicle in a circular orbit was discussed in this journal by Thomson,[1] who dealt with motions during which the spin axis of the gyro is either normal or parallel to the plane of the orbit, whereas the output axis is aligned with the local vertical, with the flight-path tangent, or with a line fixed in inertial space. The effect on stability of connecting the gimbal ring and the vehicle with a restoring spring was considered in two of the four cases treated.

The present paper extends Thomson's work in the following respects: motions not examined previously are studied, energy dissipation is taken into account, and the inertia properties of the gimbal ring are included in the analysis.

The paper is divided into two parts. The first contains a description of the system under consideration, the differential equation governing all motions of the system, and a discussion of the equilibrium solutions of this equation. In the second part, the stability of the equilibrium solutions is analyzed.

Dynamics

Figure 1 is a schematic representation of a satellite vehicle V, gimbal ring G, and rotor R in a general configuration. G and V are connected to each other by a spring-and-damper unit S. The (constant) orbital rate of the vehicle is designated by Ω, whereas ω denotes the angular speed of R relative to G. It is assumed 1) that R is driven (by means of a motor, not shown) in such a way that ω remains constant, making it possible to express ω as

$$\omega = s\Omega \tag{1}$$

where s is a dimensionless constant; and 2) that V is made to move in such a way that the angle ϕ between the flight-path tangent and the output axis, which lies in the orbit plane, is given by

$$\phi = \phi_0 + n\Omega t \tag{2}$$

where ϕ_0 and n are (dimensionless) constants. The system comprised of R and G thus possesses only one degree of freedom; and the angle θ between the normal to the orbit plane and the spin axis of the gyro serves as the associated generalized coordinate.

The inertia properties of R and G are described as follows: R has a moment of inertia J about the spin axis, and a moment of inertia I about any axis passing through the mass center of R and perpendicular to the spin axis. It is presumed that the mass center of G coincides with that of R and that the spin axis, the output axis, and a line perpendicular to both of these and passing through their point of intersection are all principal axes of inertia of G, the corresponding moments of inertia being A, B, and C, respectively. For later use, it is convenient to define two (dimensionless) inertia parameters, r and r^*, as

$$r = J/(I + B) \tag{3}$$

$$r^* = (I + C - A)/(I + B) \tag{4}$$

Since the sum of two principal moments of inertia always exceeds the third principal moment of inertia, I and J satisfy the inequality $2I > J$, whereas A, B, and C are subject to the requirement $A + B > C$. Consequently, the only values of r and r^* having physical significance are those lying in the ranges $0 < r < 2$ and $0 < r^* < 1$. It can be seen from Eq. (4) that $r^* \approx 1$ may be interpreted as representing either the case of a "light" gimbal ring (i.e., A, B, and C all negligible in comparison with I), or the case of a "thin" gimbal ring (i.e., $A + B \approx C$).

To complete the description of the system, the characteristics of the spring-and-damper unit S must be brought into evidence. It is assumed that the spring exerts a restoring torque of magnitude $c|\theta - \bar{\theta}|$, where c is a constant and $\bar{\theta}$ is the value of θ when the spring is undeformed. The (dimensionless) parameter σ, defined as

$$\sigma = c/(I + B)\Omega^2 \tag{5}$$

may be used to eliminate c whenever this appears desirable.

Finally, the damping action of S consists of a resisting torque magnitude $d|\dot{\theta}|$, where d is a constant, and the (dimensionless) parameter δ, defined as

$$\delta = d/(I + B)\Omega \tag{6}$$

characterizes this effect.

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A detailed derivation of the differential equation governing the angle θ is given in the Appendix. When primes are used to denote differentiation with respect to a (dimensionless) variable τ, defined as

$$\tau = \Omega t \qquad (7)$$

where t is the time, this differential equation is

$$\theta'' + \delta\theta' + \sigma(\theta - \bar{\theta}) + \{sr(1 + n) + (r - r^*) \times [(1 + n)^2 + 3\cos^2(\phi_0 + n\tau)]\cos\theta\}\sin\theta = 0 \qquad (8)$$

where the term that depends explicitly on τ represents the gravitational torques associated with R and G.

Any constant value $\tilde{\theta}$ of θ that satisfies Eq. (8) is called an equilibrium solution. Now, with $\theta = \tilde{\theta}$, Eq. (8) involves τ explicitly, unless $\sin\tilde{\theta} = 0$, $\cos\tilde{\theta} = 0$, $n = 0$, or $r = r^*$. At least one of these requirements must, therefore, be fulfilled in order that $\tilde{\theta}$ be an equilibrium solution, and this fact suggests the following classification.

Class 1 contains all cases for which

$$\tilde{\theta} = 0 \qquad (9)$$

Furthermore, Eq. (8) then requires that

$$\sigma\bar{\theta} = 0 \qquad (10)$$

Class 2 corresponds to

$$\tilde{\theta} = \pi/2 \qquad (11)$$

and, from Eq. (8),

$$\sigma[(\pi/2) - \bar{\theta}] + sr(1 + n) = 0 \qquad (12)$$

Class 3 consists of cases for which

$$0 < \tilde{\theta} < \pi/2 \qquad (13)$$

and

$$n = 0 \qquad (14)$$

so that, from Eq. (8),

$$\sigma(\tilde{\theta} - \bar{\theta}) + [sr + (r - r^*)(1 + 3\cos^2\phi_0)\cos\tilde{\theta}]\sin\tilde{\theta} = 0 \qquad (15)$$

Class 4 again deals with

$$0 < \tilde{\theta} < \pi/2$$

but now

$$r = r^* \qquad (16)$$

and Eq. (8) reduces to

$$\sigma(\tilde{\theta} - \bar{\theta}) + sr(1 + n)\sin\tilde{\theta} = 0 \qquad (17)$$

In the present notation, the four cases treated by Thomson can be described as follows:

Case 1

$$\tilde{\theta} = \bar{\theta} = 0 \quad n = 0 \quad \phi_0 = 0 \quad \delta = 0 \quad r^* = 1$$

Case 2

$$\tilde{\theta} = \bar{\theta} = 0 \quad n = 0 \quad \phi_0 = \pi/2 \quad \delta = 0 \quad r^* = 1$$

Case 3

$$\tilde{\theta} = \bar{\theta} = 0 \quad n = -1 \quad \delta = 0 \quad r^* = 1$$

Case 4

$$\tilde{\theta} = \bar{\theta} = \pi/2 \quad n = -1 \quad \delta = 0 \quad r^* = 1$$

The first three are seen to belong to class 1, while the fourth falls into class 2.

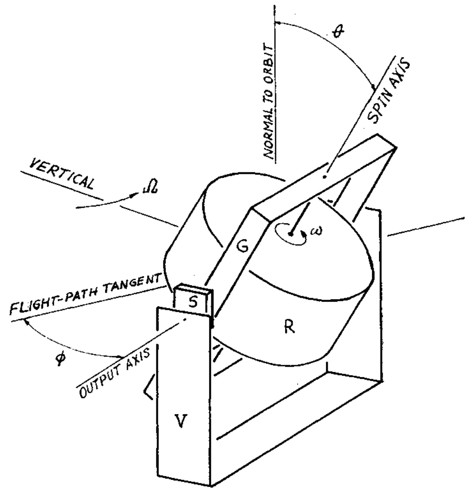

Fig. 1 Orbiting gyroscope.

Stability

To study the stability of the various equilibrium solutions, a new dependent variable y is introduced by setting

$$\theta = \tilde{\theta} + y \qquad (18)$$

in Eq. (8), all functions of y are then expanded in powers of y, first-degree terms only being retained. This gives

$$y'' + \delta y' + \{\sigma + sr(1 + n)\cos\tilde{\theta} + (r - r^*)[(1 + n)^2 + 3\cos^2(\phi_0 + n\tau)]\cos 2\tilde{\theta}\}y + \{\sigma(\tilde{\theta} - \bar{\theta}) + sr(1 + n)\sin\tilde{\theta} + (\tfrac{1}{2})(r - r^*)[(1 + n)^2 + 3\cos^2(\phi_0 + n\tau)]\sin 2\tilde{\theta}\} = 0 \qquad (19)$$

The four classes of equilibrium solutions are now characterized as follows.

Class 1: $\tilde{\theta} = 0$

As a consequence of Eqs. (9) and (10), Eq. (19) can be reduced to

$$y'' + \delta y' + \{\sigma + sr(1 + n) + (r - r^*)[(1 + n)^2 + 3\cos^2(\phi_0 + n\tau)]\}y = 0 \qquad (20)$$

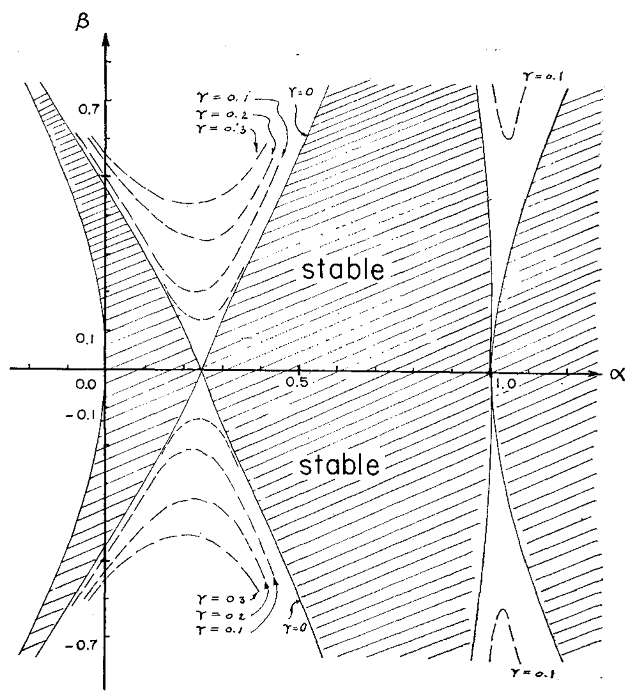

Fig. 2 Stability chart for $y'' + \gamma y' + (\alpha + \beta\cos x)y = 0$.

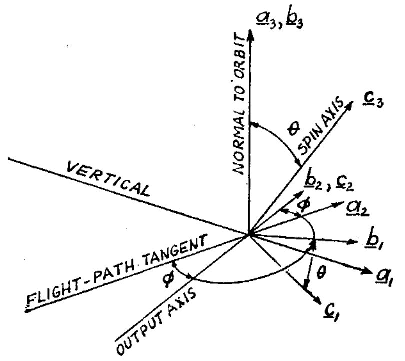

Fig. 3 Unit vectors.

Equation (20) is an equation with constant coefficients if either n or $r - r^*$ vanishes. Class 1 can thus be subdivided as follows.

Class 1a: $n = 0$

Equation (20) is now the equation of a damped harmonic oscillator, and the associated stability criterion is

$$\sigma + sr + (r - r^*)(1 + 3\cos^2\phi_0) > 0 \tag{21}$$

(With $r^* = 1$, this becomes Thomson's Eq. (5) if $\phi_0 = 0$, and Eq. (8) if $\sigma = 0$ and $\phi_0 = \pi/2$.)

Class 1b: $r = r^*$

Again, the coefficient of y in Eq. (20) must be positive for stability:

$$\sigma + sr(1 + n) > 0 \tag{22}$$

Class 1c: $n \neq 0, r \neq r^*$

Here it is convenient to introduce a new independent variable x as

$$x = 2(\phi_0 + |n|\tau) \tag{23}$$

Equation (20) then becomes

$$d^2y/dx^2 + \gamma(dy/dx) + (\alpha + \beta\cos x)y = 0 \tag{24}$$

where α, B, and γ are defined as

$$\left.\begin{aligned} \alpha &= (1/4n^2)[\sigma + sr(1 + n) + (r - r^*) \times (2.5 + 2n + n^2)] \\ \beta &= (3/8n^2)(r - r^*) \\ \gamma &= \delta/2|n| \end{aligned}\right\} \tag{25}$$

The relationship between the stability of the solutions of Eq. (24) and the parameters α, β, and γ has been investigated in detail by G. Kotowski[2] and can be displayed in the form of a stability chart, as shown in Fig. 2. The stability of a particular motion can thus be tested as soon as the system parameters have been specified. For example, suppose that the output axis is fixed in inertial space, so that $n = -1$ [see Eq. (2)] and assume that the inertia properties of the rotor and the gimbal ring are such that $r - r^* = 1$. Then $\alpha = 0.25\,\sigma + 0.375$, $\beta = 0.375$, $\gamma = 0.5\,\delta$, and Fig. 2 can be used to select various combinations of σ and δ, i.e., to "design" the spring-and-damper unit in such a way as to insure stability. For instance, with $\sigma = 0$, the point of interest in Fig. 2 is the point $\alpha = \beta = 0.375$, and it appears that $\gamma > 0.2$, and hence $\delta > 0.4$, is required for stability. Or, if damping cannot be depended upon for practical reasons, i.e., $\delta \approx 0$ and hence $\gamma \approx 0$, then Fig. 2 shows that $\alpha > 0.41$, i.e., $\sigma > 0.14$ becomes the stability condition. [Thomson's case 3 is obtained by taking $\sigma = 0$, $n = -1$, $r^* = 1$, and $\delta = 0$ in Eqs. (25).]

Class 2: $\tilde{\theta} = \pi/2$

Substitution from Eqs. (11) and (12) into Eq. (19) leads to

$$y'' + \delta y' + \{\sigma - (r - r^*)[(1 + n)^2 + 3\cos^2(\phi_0 + n\tau)]\}y = 0 \tag{26}$$

Hence there are again three subdivisions:

Class 2a: $n = 0$

For stability,

$$\sigma - (r - r^*)(1 + 3\cos^2\phi_0) > 0 \tag{27}$$

Class 2b: $r = r^*$

For stability,

$$\sigma > 0 \tag{28}$$

Class 2c: $n \neq 0, r \neq r^*$

When x [see Eq. (23)] is again used as the independent variable, Eq. (26) assumes the form of Eq. (24), and Fig. 2 furnishes the relevant stability information, provided that α, β, and γ be defined as

$$\left.\begin{aligned} \alpha &= (1/4n^2)[\sigma + (r - r^*)(-0.5 + 2n + n^2)] \\ \beta &= -(3/8n^2)(r - r^*) \\ \gamma &= \delta/2|n| \end{aligned}\right\} \tag{29}$$

(Thomson's case 4 is obtained by again taking $\sigma = 0$, $n = -1$, $r^* = 1$, and $\delta = 0$.)

Class 3: $0 < \tilde{\theta} < \pi/2$, $n = 0$

Equations (19, 14, and 15) now give

$$y'' + \delta y' + [\sigma + sr\cos\tilde{\theta} + (r - r^*) \times (1 + 3\cos^2\phi_0)\cos 2\tilde{\theta}]y = 0 \tag{30}$$

For stability, it is thus necessary that

$$\sigma + sr\cos\tilde{\theta} + (r - r^*)(1 + 3\cos^2\phi_0)\cos 2\tilde{\theta} > 0 \tag{31}$$

Class 4: $0 < \tilde{\theta} < \pi/2$, $r = r^*$

Equations (19, 16, and 17) lead to

$$y'' + \delta y' + [\sigma + sr(1 + n)\cos\tilde{\theta}]y = 0 \tag{32}$$

so that the stability condition becomes

$$\sigma + sr(1 + n)\cos\tilde{\theta} > 0 \tag{33}$$

Table 1 is intended to facilitate the use of these results. The numbers appearing there refer to equations numbered correspondingly, the first one in each box designating the relevant equilibrium condition and the second the associated stability criterion.

Appendix

Let $(\mathbf{T}^R)_g$ and $(\mathbf{T}^G)_g$ be the gravitational torques, and $(\mathbf{T}^R)_i$ and $(\mathbf{T}^G)_i$ the inertia torques for the rotor R and the gimbal ring G, and let $(\mathbf{T}^G)_c$ be the resultant moment about the mass center of the system of all contact forces exerted on G by the vehicle V. Then, in accordance with D'Alembert's Principle,

$$(\mathbf{T}^R)_g + (\mathbf{T}^G)_g + (\mathbf{T}^R)_i + (\mathbf{T}^G)_i + (\mathbf{T}^G)_c = 0 \tag{A1}$$

Table 1 Summary of results

	$\tilde{\theta} = 0$	$0 < \tilde{\theta} < \pi/2<$	$\tilde{\theta} = \pi/2$
$n = 0$	10, 21	15, 31	12, 27
$r = r^*$	10, 22	17, 33	12, 28
$n \neq 0, r \neq r^*$	10, 25, and Fig. 2		12, 29, and Fig. 2

Let $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{c}_i$, with $i = 1, 2, 3$, be three sets of unit vectors, the vectors of each set being mutually perpendicular, as indicated in Fig. 3. Then $(\mathbf{T}^G)_c$ can be expressed as

$$(\mathbf{T}^G)_c = B_1\mathbf{b}_1 - [c(\theta - \bar{\theta}) + d\dot{\theta}]\mathbf{b}_2 + B_3\mathbf{b}_3 \quad \text{(A2)}$$

where B and B_3 represent the action of the gimbal bearings. To eliminate B_1 and B_3, note that $\mathbf{c}_2$ is perpendicular to both $\mathbf{b}_1$ and $\mathbf{b}_3$, and dot-multiply Eq. (A1) with the unit vector $\mathbf{c}_2$ after using Eq. (A2):

$$\mathbf{c}_2\cdot[(\mathbf{T}^R)_g + (\mathbf{T}^G)_g + (\mathbf{T}^R)_i + (\mathbf{T}^G)_i] = c(\theta - \bar{\theta}) + d\dot{\theta} \quad \text{(A3)}$$

$(\mathbf{T}^R)_g$ and $(\mathbf{T}^G)_g$ are given by

$$(\mathbf{T}^R)_g = 3\Omega^2\mathbf{a}_1 \times (\mathbf{I}^R\cdot\mathbf{a}_1) \quad \text{(A4)}$$

$$(\mathbf{T}^G)_g = 3\Omega^2\mathbf{a}_1 \times (\mathbf{I}^G\cdot\mathbf{a}_1) \quad \text{(A5)}$$

where $\mathbf{I}^R$ and $\mathbf{I}^G$, the centroidal inertia dyadics of R and G, can be expressed as

$$\mathbf{I}^R = I\mathbf{c}_1\mathbf{c}_1 + I\mathbf{c}_2\mathbf{c}_2 + J\mathbf{c}_3\mathbf{c}_3 \quad \text{(A6)}$$

$$\mathbf{I}^G = C\mathbf{c}_1\mathbf{c}_1 + B\mathbf{c}_2\mathbf{c}_2 + A\mathbf{c}_3\mathbf{c}_3 \quad \text{(A7)}$$

the unit vectors $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ being parallel to centroidal principal axes of both bodies. Furthermore,

$$\mathbf{a}_1 = \cos\phi\cos\theta\,\mathbf{c}_1 - \sin\phi\mathbf{c}_2 + \cos\theta\sin\theta\,\mathbf{c}_3 \quad \text{(A8)}$$

Consequently,

$$\mathbf{c}_2\cdot(\mathbf{T}^R)_g = 3\Omega^2(I - J)\cos^2\phi\sin\theta\cos\theta \quad \text{(A9)}$$

[see Eqs. (A4, A6, and A8)]

and

$$\mathbf{c}_2\cdot(\mathbf{T}^G)_g = 3\Omega^2(C - A)\cos^2\phi\sin\theta\cos\theta \quad \text{(A10)}$$

[see Eqs. (A5, A7, and A8)]

The contributions of the inertia torques $(\mathbf{T}^R)_i$ and $(\mathbf{T}^G)$ to Eq. (A3) are given by

$$\mathbf{c}_2\cdot(\mathbf{T}^R)_i = -[I\alpha_2{}^G - (A - C)\omega_3{}^R\omega_1{}^R] \quad \text{(A11)}$$

$$\mathbf{c}_2\cdot(\mathbf{T}^G)_i = -[B\alpha_2{}^G - (A - C)\omega_3{}^R\omega_1{}^R] \quad \text{(A12)}$$

where the α's and ω's are angular acceleration and angular velocity measure numbers, and the subscripts refer to $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$. For R, these are found as follows:

$$\boldsymbol{\omega}^R = s\Omega\mathbf{c}_3 + \dot{\theta}\mathbf{c}_2 + (1 + n)\Omega\mathbf{b}_3 \quad \text{(A13)}$$

$$\omega_1{}^R = \boldsymbol{\omega}^R\cdot\mathbf{c}_1,\ \omega_3{}^R = \boldsymbol{\omega}^R\cdot\mathbf{c}_3 \quad \text{(A14)}$$

$$\omega_1{}^R = -(1 + n)\Omega\sin\theta \quad \text{(A15)}$$

[see Eqs. (A13) and (A14)]

$$\omega_3{}^R = s\Omega + (1 + n)\Omega\cos\theta \quad \text{(A16)}$$

[see Eqs. (A13) and (A14)]

$$\boldsymbol{\alpha}^R = (d/dt)\boldsymbol{\omega}^R = \ddot{\theta}\mathbf{c}_2 + [\dot{\theta}\mathbf{c}_2 + (1 + n)\Omega\mathbf{b}_3] \times \boldsymbol{\omega}^R \quad \text{(A17)}$$

$$\alpha_2{}^R = \boldsymbol{\alpha}^R\cdot\mathbf{c}_2 = \ddot{\theta} + s(1 + n)\Omega^2\sin\theta \quad \text{(A18)}$$

[see Eqs. (A17) and (A13)]

To find the corresponding expressions for G, it is only necessary to replace R with G and to set s equal to zero. Thus

$$\omega_1{}^G = -(1 + n)\Omega\sin\theta \qquad \text{[see Eq. (A15)]} \quad \text{(A19)}$$

$$\omega_3{}^G = (1 + n)\Omega\cos\theta \qquad \text{[see Eq. (A16)]} \quad \text{(A20)}$$

$$\alpha_2{}^G = \ddot{\theta} \qquad \text{[see Eq. (A18)]} \quad \text{(A21)}$$

and Eqs. (A11) and (A12) now lead to

$$\mathbf{c}_2\cdot(\mathbf{T}^R)_i = -I\ddot{\theta} - (1 + n)\Omega^2\sin\theta[sJ + (1 + n)(J - I)\cos\theta] \quad \text{(A22)}$$

and to

$$\mathbf{c}_2\cdot(\mathbf{T}^G)_i = -B\ddot{\theta} - (1 + n)^2\Omega^2(A - C)\sin\theta\cos\theta \quad \text{(A23)}$$

Substitution from Eqs. (A9, A10, A17, and A18) into (A3) gives

$$3\Omega^2(I - J + C - A)\cos^2\phi\sin\theta\cos\theta - (I + B)\ddot{\theta} - (1 + n)\Omega^2\sin\theta[sJ - (1 + n)(I - J + C - A)\cos\theta] = c(\theta - \bar{\theta}) + d\dot{\theta}$$

and this, after division by $(I + B)\Omega^2$ and use of Eqs. (2–6), yields

$$(\ddot{\theta}/\Omega^2) + \delta(\dot{\theta}/\Omega) + \sigma(\theta - \bar{\theta}) + \{sr(1 + n) + (r - r^*)[(1 + n)^2 + 3\cos^2(\phi_0 + n\Omega)]\cos\theta\}\sin\theta = 0$$

References

[1] Thomson, W. T., "Stability of single axis gyros in a circular orbit," AIAA J. **1**, 1556–1559 (1963).

[2] Kotowski, V. G., "Lösungen der Inhomogenen Mathieuschen Differential-gleichung Periodischer Störfunktion Beliebiger Frequenz," Z. Angew. Math. Mekh. **23**, 213–229 (1943).